INTERNATIONAL JOURNAL FOR NUMERICAL METHODS IN FLUIDS
Int. J. Numer. Meth. Fluids 2001; **35**: 39–54

Boundary domain integral method for transport phenomena in porous media

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SUMMARY

A boundary domain integral method (BDIM) for the solution of transport phenomena in porous media is presented. The complete, so-called modified Navier–Stokes equations (Brinkman-extended Darcy formulation with inertial term included) have been used to describe the fluid motion in porous media. Velocity–vorticity formulation (VVF) of the conservative equations is employed. In this paper, the proposed numerical scheme is tested on a particular case of natural convection and the results of flow and heat transfer characteristics of a fluid in a vertical porous cavity heated from the side and saturated with Newtonian fluid are presented in detail. Copyright © 2001 John Wiley & Sons, Ltd.

KEY WORDS: boundary domain integral method; Brinkman-extended Darcy formulation; natural convection; porous media; velocity–vorticity formulation

1. INTRODUCTION

Transport phenomena in porous media arise in many diverse fields of science and engineering, such as hydrology, civil and mechanical engineering, chemical and petroleum engineering. Civil engineering deals, for example, with practical problems like the flow of water in aquifers, the movement of moisture through and under engineering structures, the transport of pollutants in aquifers, and heat transport in thermal insulation.

Over the past decades, porous media have been studied both experimentally and theoretically. With the advent of precise instruments and new experimental techniques, it has become possible to measure a wide variety of physical properties of porous media and transport phenomena therein. New computational methods and technologies have also allowed us to model and simulate various phenomena in porous media, and thus a deeper understanding of these problems is being gained on a perpetual basis.

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Received July 1999
Revised December 1999

The term porous media usually refers to material consisting of a solid matrix and interconnected pores. In the present work, we assume that the solid matrix is rigid, and on account of the interconnectedness of the pores, the flow of the fluid is allowed to pass through the matrix. Transport phenomena are fluid transport processes describing how various extensive quantities, e.g. velocity, mass and heat, are transported through a porous media domain.

When the temperature of the saturated fluid phase in porous media is not uniform, flows induced by buoyancy effects occur. These flows, depending on density differences due to temperature gradients and the pertinent boundary conditions, are commonly called free or natural convection. Due to its numerous applications in energy-related engineering problems, the natural convection is gaining, over the past decade, strongly enhanced interest and has become one of the most commonly studied transport phenomena in porous media. Studies have been reported dealing with different geometries and a variety of heating conditions. For example, a vertical cavity in which a horizontal temperature gradient is induced by side walls maintained at different temperatures has been analysed [1,2]. Others have examined the natural convection in porous layers heated from below [3,4]. In all of these studies, use has been made of Brinkman-extended Darcy formulation as a governing momentum equation, because in earlier works it has been well established that the pure Darcy law does not give satisfactory results when one wants to take into account the no-slip boundary condition [5].

The numerical methods often used for the solution of governing equations, which in most cases are written in vorticity–stream function formulation, are the finite difference method (FDM) and the finite volume method (FVM). The objective of the present work is to examine the Brinkman-extended Darcy formulation with the transport term included as applied to the case of natural convection in porous cavity heated from the side, utilizing an approach based on the boundary domain integral method (BDIM) [6]. The main advantage of the proposed BDIM scheme, as compared with classical domain-type numerical techniques, is that it offers an effective way of dealing with boundary conditions on the solid walls when solving the vorticity equation. Namely, the boundary vorticity in the BDIM is computed directly from the kinematic part of the computation (as described by the Equation (4)) and not through the use of some approximate formulae.

2. GOVERNING EQUATIONS

In principle, the equations that formulate various transport phenomena in porous media are known and may be written at the microscopic level. However, at this level they can not be solved, as the geometry of the surface that bounds the phase is not observable and/or is too complex to be described adequately [7]. Another level of description is therefore needed; namely, the macroscopic level, at which the measurable, continuous and differentiable quantities may be determined and boundary value problems can be stated and later solved. The macroscopic constitutive equations for the multiphase system, called porous media, are obtained by averaging the governing microscopic equations valid for pure fluid over the representative elementary volume (REV), keeping in mind that only the distinct part of the REV (expressed with porosity ϕ) is available for the fluid flow [8]. The averaging process over

suitable REV, which has to be determined such that, irrespective of its position in porous media, it always contain both a persistent solid and a fluid phases, results in the system of governing equations describing transport phenomena in porous media which consists of

continuity equation

$$\frac{\partial v_i}{\partial x_i} = 0 \tag{1}$$

momentum equation—Brinkman equation

$$\frac{1}{\phi}\frac{\partial v_i}{\partial t} + \frac{1}{\phi^2}\frac{\partial v_j v_i}{\partial x_j} = \underbrace{-\frac{1}{\rho}\frac{\partial P}{\partial x_i} + Fg_i - \frac{\gamma}{K}v_i}_{\text{Darcy law}} + \underbrace{\frac{\bar{\gamma}}{\phi}\frac{\partial^2 v_i}{\partial x_j \partial x_j} + \frac{1}{\phi}\frac{\partial}{\partial x_j}(2\gamma'' s_{ij})}_{\text{Brinkman extension}} \tag{2}$$

energy equation

$$\sigma\frac{\partial T}{\partial t} + \frac{\partial v_j T}{\partial x_j} = \bar{a}_{\mathrm{p}}\frac{\partial^2 T}{\partial x_j \partial x_j} + \frac{\partial}{\partial x_j}\left(a''_{\mathrm{p}}\frac{\partial T}{\partial x_j}\right) \tag{3}$$

where v_i, ϕ, K are filtration velocity, porosity and permeability of porous media respectively. The vector field functions g_i and x_i represent gravity and position, while the scalar quantities $P = p - \rho g_i r_i$ and T are modified pressure and temperature. The material property ρ describes the mass density, assumed to be a constant, and s_{ij} stands for the strain rate tensor, $s_{ij} = \frac{1}{2}(\partial v_i/\partial x_j + \partial v_j/\partial x_i)$. The normalized density–temperature variation function F is written as $F = (\rho - \rho_0)/\rho_0 = -\beta_T(T - T_0)$, with ρ_0 denoting the reference mass density at temperature T_0 and β_T being the thermal volume expansion coefficient. The material property $\gamma = \mu/\rho$ is the kinematic viscosity and is partitioned into its constant and perturbated part as $\gamma = \bar{\gamma} + \gamma''$. Coefficient σ represents the ratio between the volumetric heat capacity of solid and fluid phases and reads $\sigma = \phi + (\rho_{\mathrm{s}} c_{\mathrm{s}}/\rho c)(1 - \phi)$, where ρ_{s} and c_{s} stand for mass density and specific isobaric heat of (only) the solid part of porous media. Finally, the coefficient $\bar{a}_{\mathrm{p}}$ is the constant part of thermal diffusivity and a''_{p} is the perturbated part of thermal diffusivity, such that $a_{\mathrm{p}} = \bar{a}_{\mathrm{p}} + a''_{\mathrm{p}}$, where thermal diffusivity is calculated as $a_{\mathrm{p}} = \lambda_{\mathrm{p}}/\rho c$, and λ_{p} is the heat conductivity of porous media defined as $\lambda_{\mathrm{p}} = (1 - \phi)\lambda_{\mathrm{s}} + \phi\lambda$, with λ_{s} denoting a heat conductivity of the solid.

As is known, the Brinkman extension expresses the viscous resistance or viscous drag force exerted by the solid phase on the flowing fluid at their contact surfaces. With the Brinkman equation one is able to satisfy the no-slip boundary conditions on an impermeable surface, which bounds the porous media. The novelty in our work is that the Brinkman term, as appearing in Equation (2), consists of two parts. The first part, well known in the literature, is a constant and the second one consists of the term that enables us to include the possibility of general viscosity variation in the computation, a fact of particular importance when dealing with non-Newtonian saturating fluids. It is important to stress that the Brinkman equation is essentially an interpolation scheme between the Navier–Stokes and Darcy equations. It is well

known [2–8] that, in the limit when the porosity approaches unity ($\phi \to 1$) and consequently the permeability tends towards infinity ($K \to \infty$), the Brinkman equation transforms into the classical Navier–Stokes equation for a pure fluid. Meanwhile, for the permeability converging to zero ($K \to 0$), the Brinkman term becomes negligible and the Darcy law is than recovered.

Due to the general complexity of transport phenomena in porous media we have utilized certain assumptions and suppositions as follows:

- the solid phase of the porous media is homogeneous, isotropic and non-deformable (i.e. rigid) substance;
- the fluid phase is described as an incompressible, viscous, single phase;
- porous media are saturated—meaning that the fluid occupies the entire void space;
- the two average temperatures, T_s for the solid phase and T_f for the fluid phase, are assumed to be identical in the same REV, so that the thermal behaviour of the porous media is described by a single equation for the average temperature $T \equiv T_s \equiv T_f$.

In the BDIM, the obtained set of partial differential equations (PDEs) (1–3), also called the modified Navier–Stokes equations, are further transformed by using the velocity–vorticity variables formulation (VVF) [9]. With the vorticity vector ω_i representing the curl of the velocity field

$$\omega_i = e_{ijk} \frac{\partial v_k}{\partial x_j} \tag{4}$$

where e_{ijk} is the unit permutation tensor, the fluid motion computation scheme is partitioned into its kinematic part, as given by the elliptic velocity vector equation

$$\frac{\partial^2 v_i}{\partial x_j \partial x_j} + e_{ijk} \frac{\partial \omega_k}{\partial x_j} = 0 \tag{5}$$

and its kinetic part, as provided by the parabolic–hyperbolic vorticity transport equation, obtained as curl of the momentum equation (2)

$$\frac{\partial \omega_i}{\partial \tau_v} + v_j \frac{\partial \omega_i}{\partial x_j} = \phi \bar{\gamma} \frac{\partial^2 \omega_i}{\partial x_j \partial x_j} + \phi^2 e_{ijk} g_k \frac{\partial F}{\partial x_j} + \omega_j \frac{\partial v_i}{\partial x_j} - \frac{\phi^2 \gamma}{K} \omega_i + \phi \frac{\partial}{\partial x_j} \left(\gamma'' \frac{\partial \omega_i}{\partial x_j} \right) + \frac{\partial f_{ij}}{\partial x_j} \tag{6}$$

The τ_v is so called modified vorticity time step $\tau_v = t/\phi$, introduced only as a necessary mathematical step allowing one to use the VVF principle on our momentum equation. The quantity f_{ij} in the last term of Equation (6) is defined as $f_{ij} = \phi \gamma''(\vec{\nabla} \times s_{ij})$.

To improve convergence and stability of the coupled velocity–vorticity iterative numerical scheme, the false transient approach [10] is applied to Equation (5), resulting in the following parabolic kinematic expression:

$$\frac{\partial^2 v_i}{\partial x_j \partial x_j} - \frac{1}{\alpha} \frac{\partial v_i}{\partial t} + e_{ijk} \frac{\partial \omega_k}{\partial x_j} = 0 \tag{7}$$

where α is a relaxation parameter. It is obvious that the governing velocity equation (5) is exactly satisfied only in the steady state ($t \to \infty$), i.e. when the artificial time derivative terms vanish.

The boundary conditions assigned to the elliptic kinematic velocity equation (5) are generally of the first and second kind

$$v_i = \bar{v}_i \quad \text{on } \Gamma_1; \qquad \frac{\partial v_i}{\partial x_j} n_j = \overline{\frac{\partial v_i}{\partial n}} \quad \text{on } \Gamma_2 \tag{8}$$

The Diriclet boundary conditions arise when the velocity is prescribed over the whole surface. In this case, normal derivatives of the velocity components are the unknown boundary values in the set of kinematic equations, assuming known vorticity distribution in the solution domain. Additional difficulties appear when the velocity vector is not known *a priori* over part of the surface, i.e. outflow regions. In such cases, a reasonable choice is to assume zero velocity normal flux values through the specific part of the boundary.

The most critical computation part of the kinematics is the determination of the new boundary vorticity values, which are the only proper physical boundary conditions associated with the parabolic kinetic equation (6), as written for the whole boundary

$$e_{ijk} \frac{\partial v_k}{\partial x_j} = \bar{\omega}_i \quad \text{on } \Gamma \tag{9}$$

while the vorticity normal fluxes

$$\frac{\partial \omega_i}{\partial x_j} n_j = \frac{\partial \omega_i}{\partial n} \quad \text{on } \Gamma \tag{10}$$

are the only unknown boundary values in the vorticity kinetics.

The mathematical description of the energy kinetics is completed by providing suitable natural and essential boundary conditions as well as some initial conditions

$$T = \bar{T} \quad \text{on } \Gamma_1; \qquad -\lambda_p \frac{\partial T}{\partial x_j} = \bar{q} \quad \text{on } \Gamma_2; \qquad T = \bar{T}_0 \quad \text{in } \Omega \tag{11}$$

3. BOUNDARY DOMAIN INTEGRAL EQUATIONS

3.1. General non-linear parabolic diffusion–convective equation

Consider a non-linear time dependent diffusion–convective equation for an arbitrary conservative scalar field function u (velocity, vorticity, temperature) in the form

$$\frac{\mathrm{D}u}{\mathrm{D}t} = \frac{\partial u}{\partial t} + v_j \frac{\partial u}{\partial x_j} = \frac{\partial}{\partial x_j}\left(a \frac{\partial u}{\partial x_j}\right) + I_u \tag{12}$$

where $\mathrm{D}/\mathrm{D}t$ represents the substantial or Stokes derivative and I_u is the source term. Substituting the expression for the diffusivity variation in the form of a constant $\bar{a}$ and variable part a'', so that $a = \bar{a} + a''$, Equation (12) may be partitioned into a linear and non-linear parts in the following manner:

$$\frac{\partial u}{\partial t} + v_j \frac{\partial u}{\partial x_j} = \bar{a} \frac{\partial^2 u}{\partial x_j \partial x_j} + \frac{\partial}{\partial x_j}\left(a'' \frac{\partial u}{\partial x_j}\right) + I_u \tag{13}$$

The equation represents a parabolic initial-boundary value problem; thus some boundary and initial conditions have to be known *a priori* in order to complete the mathematical description of the problem

$$u = \bar{u} \quad \text{on } \Gamma_1; \qquad -k \frac{\partial u}{\partial x_j} = \bar{q} \quad \text{on } \Gamma_2; \qquad u = \bar{u}_0 \quad \text{in } \Omega \tag{14}$$

The parameters a and k are defined according to the considered conservation laws and corresponding constitutive hypothesis.

In the transformation from PDEs to integral equations, we consider two different non-homogenous equations, namely the modified Helmholtz PDE for the kinematic and diffusion–convective PDE for the kinetic part of the computation.

By using a finite difference approximation for the time derivative of the field function, where the time increment is defined as $\Delta t = t_F - t_{F-1}$, one has

$$\frac{\partial u}{\partial t} \approx \frac{u_F - u_{F-1}}{\Delta t} \tag{15}$$

and Equation (13) can be rewritten in a non-homogenous modified Helmholtz PDE form [9], with the following corresponding integral representation:

$$\begin{aligned} c(\xi)u(\xi) + \int_\Gamma u \frac{\partial u^*}{\partial n}\, \mathrm{d}\Gamma \\ = \frac{1}{\bar{a}} \int_\Gamma \left(a \frac{\partial u}{\partial n} - u v_n\right) u^*\, \mathrm{d}\Gamma + \frac{1}{\bar{a}} \int_\Omega \left(u v_j - a'' \frac{\partial u}{\partial x_j}\right) \frac{\partial u^*}{\partial x_j}\, \mathrm{d}\Omega + \frac{1}{\bar{a}} \int_\Omega I_u u^*\, \mathrm{d}\Omega + \beta \int_\Omega u_{F-1} u^*\, \mathrm{d}\Omega \end{aligned} \tag{16}$$

where the variable u^* is the modified Helmholtz fundamental solution [11].

The most adequate and stable integral representation could be formulated by using the fundamental solution of steady diffusion–convective PDE with reaction term [12]. Since it exist only for the case of constant coefficients, the velocity field has to be decomposed into an average constant vector $\bar{v}_i$ and a perturbated vector v_i'', such that $v_i = \bar{v}_i + v_i''$. Once again, the use of a non-symmetric finite difference approximation of the time derivative permits one to rewrite Equation (13) into the non-homogenous diffusion–convective PDE [12], with the following corresponding integral formulation:

$$c(\xi)u(\xi)+\int_\Gamma u\frac{\partial U^*}{\partial n}\,\mathrm{d}\Gamma=\frac{1}{\bar{a}}\int_\Gamma\left(a\frac{\partial u}{\partial n}-uv_n\right)U^*\,\mathrm{d}\Gamma+\frac{1}{\bar{a}}\int_\Omega\left(uv_j''-a''\frac{\partial u}{\partial x_j}\right)\frac{\partial U^*}{\partial x_j}\,\mathrm{d}\Omega$$

$$+\frac{1}{\bar{a}}\int_\Omega I_u U^*\,\mathrm{d}\Omega+\beta\int_\Omega u_{F-1}U^*\,\mathrm{d}\Omega \tag{17}$$

with U^* being the product of the diffusion–convective fundamental solution and the constant part of diffusivity as $U^*=\bar{a}u^*$, where u^* is the fundamental solution of the steady diffusion–convective PDE with first-order reaction term [11].

3.2. Modified Navier–Stokes equations

The integral representation of the modified Navier–Stokes equations for the conservative field functions, i.e. velocity, vorticity and temperature, can be readily obtained following the integral statements developed above for the general transport equation (13).

As computational results in the present work are limited to the two-dimensional case, all the subsequent equations will consequently be written for the case of a planar geometry only. Considering that each component of the velocity vector v_i, Equation (7), satisfies the non-homogenous modified Helmholtz PDE subject to the corresponding boundary and initial conditions, as given by Equation (8), applying the integral formulation given by Equation (16), we obtain the boundary-domain integral statement for the planar flow *kinematics*

$$c(\xi)v_i(\xi)+\int_\Gamma v_i\frac{\partial u^*}{\partial n}\,\mathrm{d}\Gamma=\int_\Gamma\left(\frac{\partial v_i}{\partial n}+e_{ij}\omega n_j\right)u^*\,\mathrm{d}\Gamma-e_{ij}\int_\Omega\omega\frac{\partial u^*}{\partial x_j}\,\mathrm{d}\Omega+\beta\int_\Omega v_{i,F-1}u^*\,\mathrm{d}\Omega \tag{18}$$

describing the time-dependent transport of velocity field v_i in porous media. Parameter β is defined as $\beta=1/\alpha\Delta t$ and u^* is the modified Helmholtz fundamental solution [13], which takes into account the effects of geometry, time step and material properties.

Considering that the vorticity ω and temperature T, as described by Equations (6) and (3), obey the non-homogenous diffusion–convective PDE, subject to the normal, essential and initial conditions as given by Equations (9)–(11) respectively, applying the integral formulation as given by Equation (17), we obtain the boundary-domain integral statement for the planar *vorticity kinetics*

$$c(\xi)\omega(\xi)+\int_\Gamma\omega\frac{\partial U^*}{\partial n}\,\mathrm{d}\Gamma=\frac{1}{\phi\bar{\gamma}}\int_\Gamma\left(\phi\gamma\frac{\partial\omega}{\partial n}-\omega v_n+\phi^2e_{ij}n_ig_jF+f_jn_j\right)U^*\,\mathrm{d}\Gamma$$
$$+\frac{1}{\phi\bar{\gamma}}\int_\Omega\left(\omega v_j''-\phi^2e_{ij}g_jF-\phi\gamma''\frac{\partial\omega}{\partial x_j}-f_j\right)\frac{\partial U^*}{\partial x_j}\,\mathrm{d}\Omega$$
$$-\frac{1}{\phi\bar{\gamma}}\int_\Omega\frac{\phi^2\gamma}{K}\omega U^*\,\mathrm{d}\Omega+\beta\int_\Omega\omega_{F-1}U^*\,\mathrm{d}\Omega \tag{19}$$

describing the time-dependent transport of vorticity ω in the porous media domain. The parameter β is defined as $\beta = 1/\phi\bar{\gamma}\Delta\tau_v$, $U^* = \bar{\gamma}\phi u^*$ and u^* is the elliptic diffusion–convective fundamental solution of the steady diffusion–convective PDE, with first-order reaction term, considering the effects of geometry, material properties, modified vorticity time step and velocity [13]. Finally, we obtain the boundary–domain integral statement for the planar *heat energy kinetics*

$$c(\xi)T(\xi) + \int_\Gamma T\frac{\partial U^*}{\partial n}\,\mathrm{d}\Gamma = \frac{1}{\bar{a}_\mathrm{p}}\int_\Gamma \left(a_\mathrm{p}\frac{\partial T}{\partial n} - Tv_n\right)U^*\,\mathrm{d}\Gamma + \frac{1}{\bar{a}_\mathrm{p}}\int_\Omega \left(v_j'' T - a_\mathrm{p}''\frac{\partial T}{\partial x_j}\right)\frac{\partial U^*}{\partial x_j}\,\mathrm{d}\Omega + \beta\int_\Omega T_{F-1}U^*\,\mathrm{d}\Omega \tag{20}$$

describing the time-dependent transport of temperature T in porous media. Here, once again β is defined as $\beta = 1/\sigma\Delta\tau_T$ and τ_T is the modified temperature time step introduced only for the proper mathematical treatment of the heat energy equation (3), as $\tau_T = t/\sigma$. As previously, $U^* = \bar{a}_\mathrm{p}u^*$ and u^* is the fundamental solution of the steady diffusion–convective PDE with a first-order reaction term [13].

4. DISCRETIZED BOUNDARY DOMAIN INTEGRAL EQUATIONS

4.1. Formulation for general non-linear partial differential equation

Searching for an approximate numerical solution, the corresponding integral equations are written in a discretized manner [14]. The integrals over the boundary and domain are approximated by a sum of integrals over E individual boundary elements and C internal cells respectively. The variation of field functions or their products within each boundary element or internal cell is approximated by the use of appropriate interpolation polynomials. After applying the discretized integral equations to all subdomain boundary and internal nodes, the following implicit matrix systems can be obtained for the modified Helmholtz PDE:

$$[H]\{u\} = [G]\left[\frac{a}{\bar{a}}\right]\left\{\frac{\partial u}{\partial n}\right\} - \frac{1}{\bar{a}}[G][v_n]\{u\} + \frac{1}{\bar{a}}[D_j][v_j]\{u\} - [D_j]\left[\frac{a''}{\bar{a}}\right]\left\{\frac{\partial u}{\partial x_j}\right\} + \frac{1}{\bar{a}}[B]\{I_u\} + \beta[B]\{u\}_{F-1} \tag{21}$$

and for the diffusion–convective PDE:

$$[H]\{u\} = [G]\left[\frac{a}{\bar{a}}\right]\left\{\frac{\partial u}{\partial n}\right\} - \frac{1}{\bar{a}}[G][v_n]\{u\} + \frac{1}{\bar{a}}[D_j][v_j'']\{u\} - [D_j]\left[\frac{a''}{\bar{a}}\right]\left\{\frac{\partial u}{\partial x_j}\right\} + \frac{1}{\bar{a}}[B]\{I_u\} + \beta[B]\{u\}_{F-1} \tag{22}$$

To improve the economics of the computation and thus widen the applicability of the proposed numerical algorithm, the subdomain technique has been chosen [15]. The idea is to

partition the entire solution domain into subdomains to which the same discretized numerical procedure can be applied. The final system of equations for the entire domain is then obtained by adding the sets of equations for each subdomain considering the compatibility and equilibrium conditions between their interfaces, resulting in a more sparse matrix system, suitable to be solved by iterative techniques. For instance, the following conditions may be applied on the interface indicated with Γ_1 between subdomains Ω_1 and Ω_2:

$$u|_I^1 = u|_I^2, \qquad k\frac{\partial u}{\partial n}\bigg|_I^1 = -k\frac{\partial u}{\partial n}\bigg|_I^2 \tag{23}$$

The discrete model is based on a substructure technique derived to its limit version following the concept of finite volume, e.g. that each quadrilateral internal cell represents one subdomain bounded by four boundary elements. The geometrical singularities are overcome by using 3-node discontinuous quadratic boundary elements combined with 9-node corner continuous internal cells.

4.2. Formulation for modified Navier–Stokes equations

The discretized integral representations for the modified Navier–Stokes equations could be obtained by following the solution procedure as developed above for the general conservation field function u. Using the discretized equation (21), having in mind the boundary–domain integral equation (18), the following implicit matrix system is obtained for the *kinematic*:

$$[H]\{v_i\} = [G]\left\{\frac{\partial v_i}{\partial n}\right\} + e_{ij}[G]\{\omega n_j\} - e_{ij}[D_j]\{\omega\} + \beta[D]\{v_i\}_{F-1} \tag{24}$$

to be solved for unknown boundary velocity components or their normal derivatives respectively, while the computation of all internal domain velocity components, if needed, is performed in an explicit manner point by point.

Applying Equation (22) to the corresponding Equations (19) and (20), the implicit matrix system for the *vorticity kinetics*

$$\begin{aligned}[H]\{\omega\} = {} & [G]\left[\frac{\gamma}{\bar{\gamma}}\right]\left\{\frac{\partial \omega}{\partial n}\right\} - \frac{1}{\phi\bar{\gamma}}[G][v_n]\{\omega\} + \frac{1}{\phi\bar{\gamma}}[D_j][v_j'']\{\omega\} - \frac{1}{\phi\bar{\gamma}}\left[\frac{\phi^2\gamma}{K}\right][B]\{\omega\} \\ & + \frac{1}{\phi\bar{\gamma}}[G]\{\phi^2 e_{ij}n_i g_j F + f_j n_j\} - [D_j]\left[\frac{\gamma''}{\bar{\gamma}}\right]\left\{\frac{\partial \omega}{\partial x_j}\right\} - \frac{1}{\phi\bar{\gamma}}[D_j]\{\phi^2 e_{ij}g_j F + f_j\} \\ & + \beta[B]\{\omega\}_{F-1}\end{aligned} \tag{25}$$

is obtained to be solved for unknown boundary vorticity flux values and unknown domain vorticity values, while the implicit matrix system for *heat energy kinetics*

$$[H]\{T\} = [G]\left[\frac{a_p}{\bar{a}_p}\right]\left\{\frac{\partial T}{\partial n}\right\} - \frac{1}{\bar{a}_p}[G][v_n]\{T\} + \frac{1}{\bar{a}_p}[D_j][v_j'']\{T\} - [D_j]\left[\frac{a_p''}{\bar{a}_p}\right]\left\{\frac{\partial T}{\partial x_j}\right\} + \beta[B]\{T\}_{F-1} \tag{26}$$

is recovered in order to determine the unknown boundary temperature flux or boundary temperature values and temperature internal domain values.

5. SOLUTION PROCEDURE

The kinematics given by Equation (24) and the velocity boundary conditions prescribed by Equation (8) cannot assure a solenoidality of the velocity field for an arbitrary vorticity distribution, so this property may be fulfilled only by coupling kinetic and kinematic equations. Thus, the solenoidality conditions of the velocity and vorticity field requires a coupled iterative solution of the non-linear dynamic system as given by Equations (24)–(26) with the corresponding boundary conditions described by Equations (9)–(11). To obtain a solution of the fluid motion problem, the following iterative steps have to be performed.

1. Start with some initial values for the vorticity distribution.
2. Kinematic computational part:
   - solves implicit sets for boundary velocity or velocity normal flux values—Equation (24),
   - transforms new function values from element nodes to cell nodes,
   - computes the gradient of the velocity components,
   - determines new boundary vorticity values—Equation (9),
   - determines new boundary domain integral kinetic matrices, if the constant velocity vector is perturbed more than the prescribed tolerance.
3. Energy kinetic computational part:
   - solves implicit set for boundary and domain values—Equation (26),
   - transforms new function values from element nodes to cell nodes.
4. Vorticity kinetic computational part:
   - solves implicit set for unknown boundary vorticity flux and internal domain vorticity values—Equation (25),
   - transforms new function values from element nodes to cell nodes.
5. Relaxation of all new values and the convergence examination. If the convergence criterion is satisfied, then stop; otherwise go to step 2.

6. NATURAL CONVECTION IN POROUS CAVITY

To check the validity of proposed numerical procedure we will discuss the problem of natural convection in a vertical porous cavity. The description of the physical problem is shown on Figure 1 and represents a two-dimensional, vertical cavity filled with an isotropic, homogeneous, Newtonian fluid-saturated porous media. One vertical wall of the cavity is isothermally heated, the other is isothermally cooled, and the horizontal walls are adiabatic.

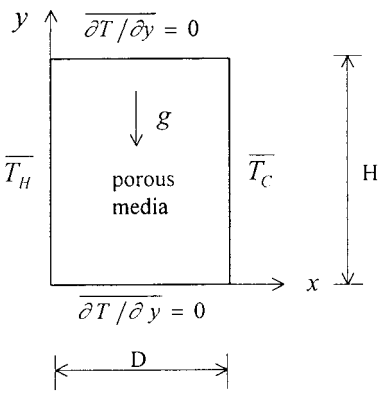

Figure 1. Geometry and boundary conditions for porous cavity.

The thermo-physical properties of the solid and the fluid are assumed to be constant except for the density variation in the body force term. Assuming that the solid particles and the fluid are in thermal equilibrium, the governing equations are written in the form of Equations (1)–(3). The computations have been carried out for the complete Brinkman-extended Darcy model with the transport term in the momentum equation included. Whenever we consider the Brinkman term, we have to deal with a parameter called the Darcy number, *Da* [1], appearing as the ratio between the permeability and the characteristic length multiplied with the viscosity ratio Λ, which is, in our case, equal to the reciprocal value of porosity ($\Lambda = 1/\phi$). We must stress that, with the use of the BDIM, the Darcy number is not explicitly derived as we are not employing the non-dimensionalized formulation of the governing equations, which is the common procedure used with other numerical methods. We will use that parameter only for the reason of comparison with the published results, noting that the permeability itself completely defines the characteristics of porous media when the BDIM is used. Thus, the governing parameters for the present problem are

- porosity ϕ,
- modified (porous) Rayleigh number $Ra^* = g\beta KD\Delta T/\gamma a_p$,
- permeability of porous media K, defined in the terms of the so-called Darcy number as $Da = (1/\phi)(K/D^2)$,
- aspect ratio $A = H/D$,
- ratio between volumetric heat capacity of solid and fluid phase defined as the so-called heat capacity ratio $\sigma = \phi + (\rho_s c_s/\rho c)(1-\phi)$,

where D, H, ΔT are the width of the cavity, the height of the cavity and the temperature difference between hot and cold walls respectively. Parameter β is the isobaric coefficient of thermal expansion of the fluid.

We have tested our numerical model on several different cases and therefore we can confirm the findings of others [1,2,5], that the effect of an increase in the Darcy number appears to be

very similar at all modified Rayleigh numbers, although it is known that the effect of the viscous (Brinkman) term becomes more important at high modified Rayleigh numbers. The proposed BDIM scheme has been verified in a square cavity with aspect ratio $A = 1$ and because of the above-mentioned similarity we graphically present only one example, for $Ra^* = 500$, in order to outline the relevant characteristics that are common to all modified Rayleigh numbers.

The boundary conditions for computed test examples are

$$\begin{aligned}
&\bar{v}_x = \bar{v}_y = 0 && \text{for } x = 0, D \quad \text{and} \quad y = 0, H \\
&\frac{\partial \bar{T}}{\partial y} = 0 && \text{for } y = 0, H \\
&\bar{T} = \bar{T}_{\mathrm{H}} = 0.5 && \text{for } x = 0 \\
&\bar{T} = \bar{T}_{\mathrm{C}} = -0.5 && \text{for } x = D
\end{aligned}$$

The streamlines and isotherms for a square cavity with aspect ratio $A = 1$, modified Rayleigh number $Ra^* = 500$ and different Darcy numbers $Da = 10^{-1}$, 10^{-2}, 10^{-3}, 10^{-4}, are presented

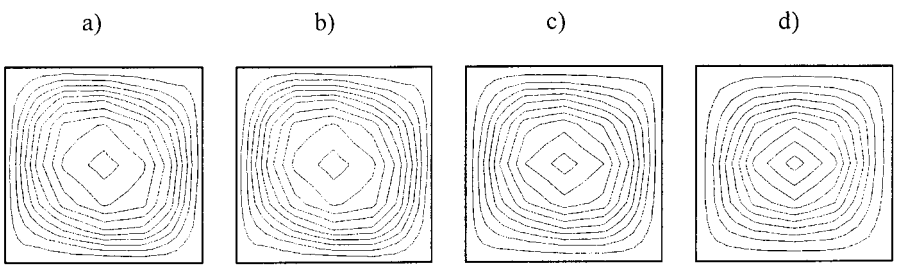

Figure 2. Streamlines for $A = 1$, $\phi = 0.5$, $\Delta T = 1$, $Ra^* = 500$; (a) $Da = 10^{-4}$, (b) $Da = 10^{-3}$, (c) $Da = 10^{-2}$, (d) $Da = 10^{-1}$.

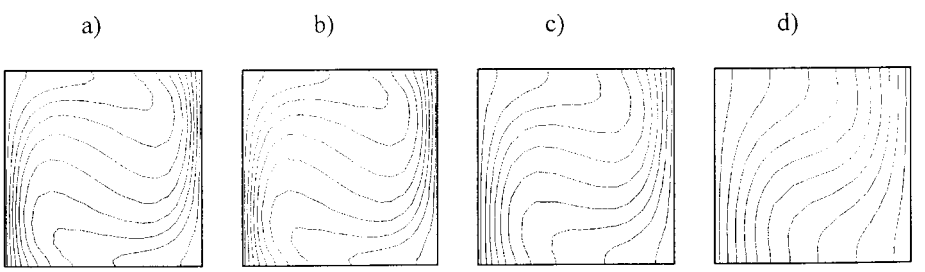

Figure 3. Isotherms for $A = 1$, $\phi = 0.5$, $\Delta T = 1$, $Ra^* = 500$; (a) $Da = 10^{-4}$, (b) $Da = 10^{-3}$, (c) $Da = 10^{-2}$, (d) $Da = 10^{-1}$.

in Figures 2 and 3. A computational mesh of 10×10 subdomains is used. Time steps ranged from $\Delta t = 10^{16}$ (steady state) for $Da = 10^{-1}$, $\Delta t = 10^{-1}$ for $Da = 10^{-2}$, $\Delta t = 10^{-2}$ for $Da = 10^{-3}$ to $\Delta t = 10^{-3}$ for $Da = 10^{-4}$ and the convergence criterion was selected as $\varepsilon = 0.00001$. The porosity is equal to $\phi = 0.5$ and the heat capacity ratio to $\sigma = 1$.

The streamlines in Figure 2(a) are observed to be closely spaced near the solid boundaries. This configuration indicates that the fluid velocity reaches a maximum near the boundaries as expected, since in the limit when $Da = 0$ (Darcy law), the velocity has a maximum on the boundaries. In this case, Da is small enough so that the viscous term, which is responsible for the boundary effects, becomes negligible and the Darcy law correctly describes the flow behaviour. Figure 2(b)–(d) illustrates typical results obtained on the basis of Brinkman model for various values of Da. It is evident that when the Darcy number increased, the boundary effects on the flow field become significant and the streamlines are observed to become relatively more sparsely spaced near the solid boundaries. This is so due to the fact that the viscous term (Brinkman term) becomes gradually more important and slows down the fluid in the neighbourhood of the solid walls. It is also observed that the region where the flow attains maximum velocity, as indicated by closely spaced streamlines, moves away from the walls towards the core region as Da is increased.

Similarly, we can observe the effects of Darcy numbers on the isotherms or on the temperature field (Figure 3). When the Darcy number is small—Figure 3(a)—the convective motion inside the cavity is strong and the isotherms are considerably distorted. The flat isotherms in the core indicate a negligible lateral conduction. As Da is increased, the viscous effects become more important and slow down the buoyancy induced flow inside the cavity. The isotherm profiles become more linear and heat transfer across the cavity results from the combined action of conduction and convection.

From the above-presented results we can clearly observe that the streamlines and isotherms redistribution are almost identical for $Da = 10^{-4}$ and $Da = 10^{-3}$, but with a further increase in the Darcy number (that means with an increase in permeability K), the velocity and temperature fields are starting to becomes significantly modified.

In what follows we will compare our results with the findings of others for which the work of Lauriat and Prasad [1] has been chosen. The rate of heat transfer expressed with the average Nusselt number $Nu = \int_0^1 \partial T/\partial n \, \mathrm{d}y$ for different modified Rayleigh numbers are collected in Table I, where in brackets the results from Reference [1] are presented. The straightforward comparison is not fully possible because, in above-mentioned study, the authors have calculated the Nusselt number considering the Brinkman momentum equation in which the transport term had been assumed to be negligible, while in our work the computations have been made on the basis of the complete Brinkman equation in the form as given by Equation (2). Also we employ the viscosity ratio Λ equal to the reciprocal value of porosity [8], while Lauriat and Prasad have taken this ratio equal to unity. Therefore, their results can be used by replacing the modified Rayleigh number Ra^* in their formulation with Ra^* multiplied by factor 2 ($\phi^{-1} = 0.5^{-1} = 2$).

The average Nusselt number is presented in Figures 4 and 5 for $A = 1$, modified Rayleigh numbers $Ra^* = 100$, 200, 500 and 1000, and different Darcy Numbers $10^{-5} \leq Da \leq 10^{-1}$. As expected, the Nusselt number approaches the conduction value ($Nu = 1$) when Ra^* approaches zero ($Ra^* \to 0$). Also, the Nusselt number always increases with Ra^*, but the effect of the

Table I. Average *Nu* numbers (values in brackets after Lauriat and Prasad [1]).

Da/Ra^*	100	200	500	1000
10^{-1}	1.026	1.061	1.370	1.815
	—	—	—	—
10^{-2}	1.479	2.016	2.823	3.691
	(1.46)	(1.70)	(2.58)	(3.30)
10^{-3}	1.816	2.666	4.030	7.410
	(1.88)	(2.41)	(3.80)	(5.42)
10^{-4}	1.895	2.718	4.370	7.921
	(2.14)	(2.84)	(4.87)	(7.37)
10^{-5}	2.010	2.765	4.474	8.200
	(2.15)	(3.02)	(5.37)	(8.41)

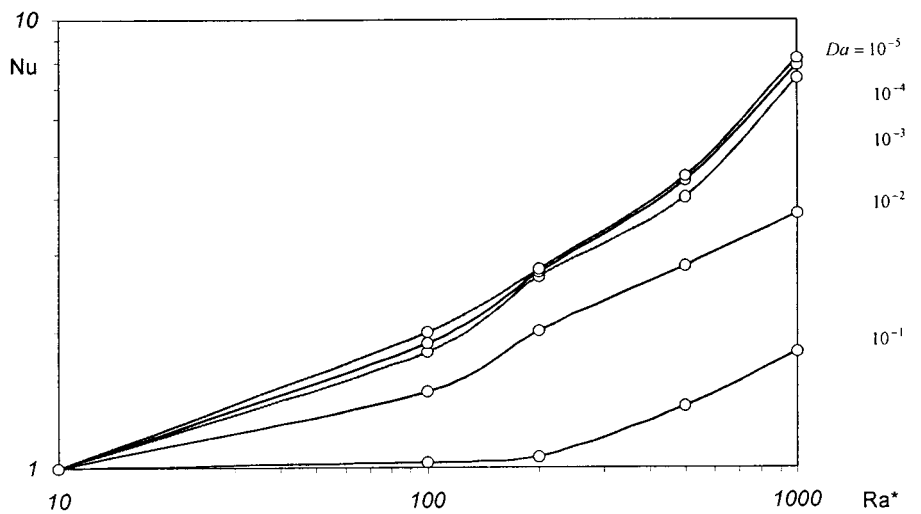

Figure 4. Average Nusselt number for $100 < Ra^* < 1000$ and $10^{-5} \leq Da \leq 10^{-1}$.

Darcy number is just the reverse. From Figure 5 we can clearly observe that the effect of the viscous Brinkman term becomes negligible when $Da < 10^{-3}$, which is in complete accord with the observations of others that solved the same problem utilizing, however, different numerical methods (for example the FDM in Reference [1]).

Using the BDIM we are therefore in position of confirming the basic fact which states that, when considering the problem of natural convection in configurations bounded by a solid wall, the Brinkman momentum equation ought to be used. The Brinkman equation namely satisfies the non-slip boundary condition on the impermeable walls that bound the porous media domain and gives physically more realistic results (especially when Darcy number is beyond 10^{-3}) than if the classical Darcy law is used as a starting momentum equation.

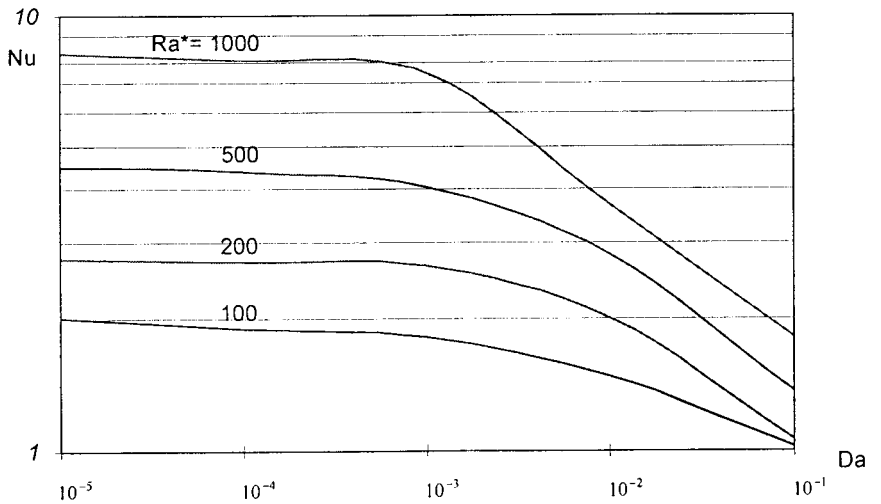

Figure 5. Effect of Darcy number on the average Nusselt number.

7. CONCLUSION

A numerical approach based on the BDIM is being applied for the solution of the problem of natural convection in porous cavity. The solution is based on the VVF of the modified Navier–Stokes equations, which allows the separation of the computational scheme into its kinematic and kinetic part respectively. The elliptic modified Helmholtz fundamental solution is used for the kinematic part of computation, while the elliptic diffusion–convective fundamental solution is employed for the kinetic one. The limited version of the subdomain technique, e.g. each subdomain is being constructed of four discontinuous 3-node quadratic boundary elements and one continuous 9-node corner continuous quadratic cell, has been applied. The proposed numerical procedure is studied, presented and discussed for the case of natural convection in porous cavity heated from the side for different Rayleigh and Darcy numbers. The very encouraging results obtained serve as a strong indication that the BDIM possesses the potential to become a powerful alternative to the existing numerical methods for solving certain class of transport phenomena in porous media.

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